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Stability of conducting viscous film flowing down an inclined plane with linear temperature variation in the presence of a uniform normal electric field

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ABSTRACT

Weakly nonlinear stability analysis of a thin liquid film falling down a heated inclined plane with linear temperature variation in the presence of a uniform normal electric field has been investigated within the finite amplitude regime. A generalized kinematic equation for the development of free surface is derived by using long wave expansion method. A normal mode approach and the method of multiple scales are used to investigate the linear and weakly nonlinear stability analysis of film flow, respectively. It is found that both Marangoni and electric Weber numbers have destabilizing effect on the film flow. The study reveals that both supercritical stability and subcritical instability are possible for this type of film flow. It is interesting to note that both the Marangoni and electric Weber numbers have qualitatively same influence on the stability characteristics but the effect of Marangoni number is much stronger compare to the electric Weber number. Scrutinizing the effect of Marangoni and electric Weber numbers on the amplitude and speed of waves it is found that, in the supercritical region amplitude and speed of the nonlinear waves increases with the increase in Marangoni and electric Weber numbers, while in the subcritical region the threshold amplitude decreases with the increase in Marangoni and electric Weber numbers. Finally, we obtain that spatially uniform solution is side-band stable in the supercritical region for our considered parameter range.

1. Introduction

The effect of electric field on a thin liquid film produces a class of problems that has attracted much attention of several researchers due to its technological applications. The presence of electric field introduces additional physical effects on the flow dynamics such as body force due to a current in conducting fluids and the Maxwell stress at the free interfaces. In the industrial process electric field has been used to destabilize the liquid film on the plane. Kim et al. [1] studied the interaction of an electrostatic field on the film flowing in an inclined plane. They found that, in the thin film limit, the effect of electric field occurred in the wave evolution as an external pressure distribution, which causes destabilizing effect on the film. It is interesting to note that, they provide a discussion on the application of their results to a proposed electrostatic liquid film space radiator. Nonlinear stability of a perfectly conducting film flowing down an inclined plane in presence of normal electric field was investigated by González and Castellanos [2]. They derived a nonlinear evolution equation with a Hilbert transform type of term within the limit of small Reynolds number and predict the destabilizing effect of the electric field in the finite amplitude regime. Recently, Mukhopadhyay and Dandapat [3] extended the study of González and Castellanos [2] within the regime of large Reynolds number and confirmed the existence of subcritical unstable and supercritical stable zones. They have also determined a critical value of the electric parameters below which the flow remains stable.

Another set of problems concerns the thermocapillary effect on the falling film down an inclined plane. The interfacial stress generated by the surface tension gradient (Marangoni effect) and the associated modes of instability are known as thermocapillary instability. Gousiss and Kelly [4] investigated the effect of thermocapillarity on a liquid film falling down an inclined uniformly heated plane by performing a linear stability analysis based on Orr–Somerfield and linearized energy equation. They found that a heated wall has a destabilizing effect on the free surface but a cooled wall stabilizes the flow. Later Miladinova et al. [5] studied the effect of non-uniform heating of the plane in the finite amplitude regime by long wave expansion method, as a consequence their study is valid only at a small vicinity of the critical Reynolds number. To overcome this limitation Kalliadasis et al. [6] have studied the problem by integral boundary layer method but they have considered the plane to be uniformly heated. This integral boundary layer method has an inherent error as it does not accurately predict the behaviour of the film close to criticality, such as the order of 20% for the critical Reynolds number. Ruyer-quil et al. [7] studied the problem by higher-order weighted residual

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approach with polynomial expansions for both velocity and temperature field to overcome the limitation about the criticality of the above problem. In spite of some limitation each study has a great importance in its own to accelerate the ongoing research in the respective field. Recently, Mukhopadhyay and Mukhopadhyay [8] investigated the influence of thermocapillarity on the span of supercritical/subcritical regimes and showed that it has a strong effect on the amplitude and speed of the nonlinear waves. The review of the literature confirms that, there is no such study addressing the effect of the above two types of problems simultaneously. In the present study, an attempt is made to consider the combined effect of uniform normal electric field that at infinity on the flow of conducting viscous film on an inclined heated plane with linear temperature variation.

2. Formulation of the problem

Consider a layer of a conducting thin liquid film flows down an inclined heated plane of inclination θ with the horizon under the action of gravity and an uniform electric field that at infinity and perpendicular to the unperturbed interface. The co-ordinate system is chosen such that x-axis along the flow and z-axis normal to the inclined plane. We assume that the electrical permittivities are constant but take different value in different medium. Due to constant permittivity, the fluid is not coupled to the electric field in the bulk (Melcher and Taylor [9]). However, electrical parameters suffer discontinuities at the interfacial region only, so interface experiences the effect of electric field. The governing equations consist of the continuity equation, Navier–Stokes equation for the flow of the liquid layer, energy equation for the temperature field and Laplace equation for the electric field. The governing equations in dimensional form can be written as:

$$\nabla\cdot\mathbf{v}=0 \quad (1)$$

$$\rho(\mathbf{v}_t+(\mathbf{v}\cdot\nabla)\mathbf{v})=-\nabla p+\rho\nu\nabla^2\mathbf{v}+\rho\mathbf{g} \quad (2)$$

$$\partial_t T+(\mathbf{v}\cdot\nabla)T=\kappa\nabla^2 T, \quad (3)$$

$$\nabla^2\Phi=0 \quad (4)$$

where $\mathbf{v}=(u,0,v)$ is the velocity vector, $\mathbf{g}=(g\sin\theta,0,-g\cos\theta)$ is the acceleration due to gravity vector and p, ρ, ν, T denote the pressure, density, kinematic viscosity and absolute temperature, respectively, and $\nabla=(\partial/\partial x,0,\partial/\partial z)$. Also $\kappa=k_T/(\rho C_p)$ denote the thermal diffusivity, k_T thermal conductivity, C_p the specific heat at constant pressure of the fluid and Φ denotes the electric potential. The pertinent boundary conditions on the inclined plane ($z=0$) and at the free surface ($z=h(x,t)$) are:

No-slip condition at the plane:

$$\mathbf{v}=0 \quad \text{at} \quad z=0, \quad (5)$$

law of temperature variation of the plane:

$$T=T_g+\mathscr{A}x \quad \text{at} \quad z=0, \quad (6)$$

kinematic boundary condition:

$$\partial_t h+(\mathbf{v}\cdot\nabla)(h-z)=0 \quad \text{at} \quad z=h, \quad (7)$$

condition that the liquid is grounded perfect conductor:

$$\Phi=0 \quad \text{at} \quad z=h, \quad (8)$$

continuity of the shear stress:

$$[[\mathbf{n}\cdot\tau\cdot\mathbf{t}]]=\nabla\sigma(T)\cdot\mathbf{t} \quad \text{at} \quad z=h, \quad (9)$$

continuity of the normal stress:

$$[[\mathbf{n}\cdot\tau\cdot\mathbf{n}]]-[[p]]=-\sigma(T)\nabla\cdot\mathbf{n} \quad \text{at} \quad z=h, \quad (10)$$

Newton's law of cooling:

$$k_T\nabla T\cdot\mathbf{n}+k_g(T-T_g)=0 \quad \text{at} \quad z=h, \quad (11)$$

where T_g denotes the temperature in the gas phase, $\mathscr{A}=(T_H-T_C)/l_0$, where T_H and T_C denote the temperatures at hotter part and the colder part, respectively, along the inclined plane and l_0 the characteristic longitudinal length scale whose order may be considered same as the wave length λ. In this study, we have taken the temperature T is increasing in the stream-wise direction and hence $\mathscr{A}$ is positive. Also $\sigma(T)$ is the surface tension of the liquid, k_g is the heat transfer coefficient between the liquid and air and [[*]] denotes a jump in the quantity as the interface is crossed from the liquid to vacuum region. $\mathbf{n}$ and $\mathbf{t}$ are the normal and tangent vectors pointing outward to the interface, respectively, and the stress tensor $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau}=\boldsymbol{\tau}^f+\boldsymbol{\tau}^e$$

where the viscous stress tensor

$$\tau^f_{ij}=\rho\nu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

and the electrical (Maxwell) stress tensor

$$\tau^e_{ij}=\epsilon_m\left[E_iE_j-\frac{1}{2}E_kE_k\delta_{ij}\right],$$

where ϵ_m is the permittivity of the concerned medium.

Further we have uniform normal electric field far from the surface, which gives

$$\Phi_z\to E_0 \quad \text{as} \quad z\to\infty, \quad (12)$$

where E_0 is the basic uniform normal applied electric field and at the free surface $(z=h(x,t))$ as another boundary condition.

The above equations are quite general regarding various coefficients (k_T, κ, μ, σ, etc.). It is well known that temperature variation in the fluid can cause dramatic changes in the above coefficients, but approximations can be made depending on the type of the problem being examined. In the foregoing analysis, we have assumed the variation of surface tension as

$$\sigma(T)=\sigma_0-\gamma(T-T_g), \quad (13)$$

where σ_0 is the surface tension at T_g, the temperature in the gas phase, which is taken as the reference temperature and $\gamma=-\partial\sigma/\partial T|_{T=T_g}$ is a positive constant for most common fluids. The assumption of linear variation of surface tension with temperature is very much compatible with the experimental data. Apart from water [8,10] there are many liquids [11] which follow the linear variation of surface tension with the temperature scales. For example, the molten tin (Sn) in the range of 520–1670 K and molten zirconium (Zr) in the range 2000–2250 K follow the law

$$\sigma(T)=561.6-0.103(T-505)\ \text{m Nm}^{-1} \quad \text{and}$$

$$\sigma(T)=1543-0.66(T-2128)\ \text{m Nm}^{-1},$$

respectively. It is to be noted here that both the liquids are highly conducting.

To express the governing equations and boundary conditions in non-dimensional form, we shall assume two length scales l_0 and h_0 as the characteristic measure for the length in longitudinal and transverse direction, respectively; l_0 may be assumed as one wave length and h_0 is the mean depth of the film, which gives $l_0\gg h_0$. Further to measure the transverse length in the vacuum, which extend to infinity, h_0 is not a proper scale, so we shall consider l_0 as the measure of the transverse length in the vacuum. The Nusselt velocity $u_0=gh_0^2\sin\theta/3\nu$ will be assumed as the characteristic velocity along the longitudinal direction. We define the dimensionless quantities as

$$x=l_0x^*,\quad h=h_0h^*,\quad z=h_0z^*\ \text{(in liquid)},$$
$$z=(l_0/h_0)\zeta\ \text{(in vacuum)},\quad t=(l_0/u_0)t^*,\quad u=u_0u^*,$$
$$v=(h_0/l_0)u_0v^*,\quad p=\rho u_0^2p^*,\quad T=T_g+T^*(T_H-T_C),$$
$$\Phi=E_0h_0\Phi^*. \quad (14)$$

Using the above dimensionless quantities in the governing equations (1)–(4) and in the boundary conditions (5)–(12) after using the relation (13) reduces to the form, after dropping the asterisk as:

(I) Equations in the liquid ($0 < z < h$)

$$u_x + v_z = 0, \tag{15}$$

$$u_t + uu_x + vu_z = -p_x + \frac{\sin\theta}{\varepsilon Fr} + \frac{1}{\varepsilon Re}(\varepsilon^2 u_{xx} + u_{zz}), \tag{16}$$

$$\varepsilon^2(v_t + uv_x + vv_z) = -p_z - \frac{\cos\theta}{Fr} + \frac{\varepsilon}{Re}(\varepsilon^2 v_{xx} + v_{zz}). \tag{17}$$

$$\varepsilon Re P_r(T_t + uT_x + vT_z) = \varepsilon^2 T_{xx} + T_{zz}. \tag{18}$$

(II) Equation in the vacuum ($\varepsilon h < \zeta < \infty$)

$$\phi_{xx} + \phi_{\zeta\zeta} = 0. \tag{19}$$

(III) Boundary conditions at the wall ($z = 0$)

$$u = 0, \qquad v = 0, \qquad T = x. \tag{20}$$

(IV) Boundary conditions at the free surface ($z = h, \zeta = \varepsilon h$)

$$v = h_t + uh_x, \tag{21}$$

$$\phi = 1 - h, \tag{22}$$

$$\left[(1-\varepsilon^2 h_x^2)(u_z + \varepsilon^2 v_x) + 4\varepsilon^2 h_x v_z\right]\left(1+\varepsilon^2 h_x^2\right)^{-1/2} = -Mn(T_x + h_x T_z), \tag{23}$$

$$p_a - p + \frac{2\varepsilon}{Re}\left(v_z\frac{1-\varepsilon^2 h_x^2}{1+\varepsilon^2 h_x^2} - h_x\frac{u_z+\varepsilon^2 v_x}{1+\varepsilon^2 h_x^2}\right) = \varepsilon^2 We(1 - Ca\cdot T)\frac{h_{xx}}{(1+\varepsilon^2 h_x^2)^{3/2}} + \frac{1}{2}E_w(1+\varepsilon^2 h_x^2)(1+\varepsilon\phi_\zeta)^2, \tag{24}$$

$$(T_z - \varepsilon^2 h_x T_x)(1+\varepsilon^2 h_x^2)^{-1/2} + Bi\cdot T = 0. \tag{25}$$

(V) Boundary condition at infinity ($\zeta \to \infty$):

$$\phi_\zeta \to 0, \tag{26}$$

where ϕ is the perturbed electric potential, i.e. $\Phi = E_0(z - h_0) + \phi$, $Fr(\equiv u_0^2/gh_0)$ is the Froude number, $Re(\equiv u_0 h_0/\nu)$ is the Reynolds number, $Pr(\equiv \nu/\kappa)$ is the Prandtl number, $Mn(\equiv 3\gamma(T_H - T_C)/\rho g l_0 h_0 \sin\theta)$ is the Marangoni number,[1] $We(\equiv \sigma_0/\rho u_0^2 h_0)$ is the Weber number, $Ca(\equiv \gamma(T_H - T_C)/\sigma_0)$ is the capillary number, $E_w(\equiv \epsilon_0 E_0^2/\rho u_0^2)$ is the electric Weber number ($\epsilon_0 = 8.854 \times 10^{-12}$ farads/m is the permittivity of the vacuum), $Bi(\equiv k_g h_0/k_T)$ is the Biot number and $\varepsilon(\equiv h_0/l_0)$ is the aspect ratio for long wave expansion. In deriving the above sets of equations we have used that the Maxwell stress vanishes inside the liquid layer while the viscous stress are null in the air. Also the dynamic influence of the air above the liquid film is ignored.

2.1. Nonlinear evolution equation

Eq. (19) is solved using (22) and (26), which gives

$$\phi = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{z(h(x')-1)}{z^2+(x-x')^2}\,dx'. \tag{27}$$

Solving Eq. (17) for lowest order, by using (24) and (27), we have

$$p_0 = p_a + \frac{3\cot\theta}{R}(h-z) + \frac{1}{2}E_w(-1 + 2\varepsilon(\mathscr{H}(h-1))_x) - \varepsilon^2 We h_{xx}, \tag{28}$$

where $\mathscr{H}(h)$ represents the Hilbert transform operator given by

$$\mathscr{H}(h(x)) = \frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{h(x')}{x'-x}\,dx',$$

with $\mathscr{P}$ the principal value of the integral. It can be shown by using the definition of the Nusselt velocity that

$$\frac{\sin\theta}{Fr} = \frac{3}{Re}. \tag{29}$$

We have assumed that the Reynolds number is O(1) and Weber number is $O(1/\varepsilon^2)$, Marangoni number is of order O(1), Capillary number is $O(\varepsilon)$, the Prandtl number is O(1), Biot number is $O(\varepsilon^2)$ and the electric Weber number is $O(1/\varepsilon)$. We are now interested in yielding a nonlinear evolution equation in terms of film thickness $h(x,t)$. Expanding the velocity components, pressure and the temperature in powers of ε as:

$$u = u_0 + \varepsilon u_1 + \cdots, \quad v = v_0 + \varepsilon v_1 + \cdots, \quad p = p_0 + \varepsilon p_1 + \cdots \quad \text{and} \quad T = T_0 + \varepsilon T_1 + \cdots$$

and substituting the above into the governing equations (15), (16) and (18) and the boundary conditions (20), (23) and (25) and also using (28) we obtain a set of PDEs for a different order of ε. The velocity profile and its depth averaged velocities for the zeroth-order and the first-order problem is obtained as

At O(1):

$$u_0 = 3\left(hz - \frac{z^2}{2}\right) - Mn\cdot z, \tag{30}$$

$$\overline{u}_0 = h^2 - \frac{1}{2}Mn\cdot h, \tag{31}$$

and at $O(\epsilon)$:

$$u_1 = Re\left[\left(\frac{3\cot\theta}{Re}h_x - \varepsilon^2 We h_{xxx} + \varepsilon E_w(\mathscr{H}(h-1))_{xx}\right)\left(\frac{z^2}{2} - hz\right) + \frac{1}{2}(3h - Mn)\left(\frac{z^4}{4} + 2h^3 z\right)h_x + \frac{1}{2}h_t z^3 + \frac{1}{2}Pr\cdot Mn(5h - 2Mn)h^2 z h_x\right], \tag{32}$$

$$\overline{u}_1 = Re\left[\left(-\frac{\cot\theta}{Re}h_x + \frac{1}{3}\varepsilon^2 We h_{xxx} - \frac{1}{3}\varepsilon E_w(\mathscr{H}(h-1))_{xx}\right)h^2 + \frac{2}{5}(3h - Mn)h^4 h_x + \frac{1}{4}Pr\cdot Mn(5h-2Mn)h^3 h_x\right]. \tag{33}$$

Integrating the continuity equation (15) with respect to z from 0 to h by using Leibnitz's rule and boundary conditions (20) and (21), we have

$$\frac{\partial h}{\partial t} + \frac{\partial(\overline{u}_0 h)}{\partial x} + \varepsilon\frac{\partial(\overline{u}_1 h)}{\partial x} + O(\varepsilon^2) = 0. \tag{34}$$

Substituting $\overline{u}_0$ and $\overline{u}_1$ from (31) and (33) in (34), we get

$$h_t + A(h)h_x + \varepsilon\left(B(h)h_x + \varepsilon C(h)(\mathscr{H}(h-1))_{xx} + \varepsilon^2 D(h)h_{xxx}\right)_x = 0, \tag{35}$$

where the suffixes denote the differentiation with respect to the corresponding variables and

$$A(h) = 3h^2 - Mnh,$$

$$B(h) = -\cot\theta h^3 + \frac{2}{5}Re(3h - Mn)h^5 + \frac{1}{4}Re\,Pr\,Mn(5h - 2Mn)h^4,$$

$$C(h) = -\frac{1}{3}ReE_w h^3,$$

$$D(h) = \frac{1}{3}ReWeh^3.$$

[1] $Mn = \varepsilon Ma/RePr$, $Ca = CrMa$, where $Ma = \gamma(T_H - T_C)h_0/\mu\kappa$ and $Cr = \mu\kappa/\sigma_0 h_0$ are, respectively, the Marangoni numbers and Crispation/capillary in usual definition.

3. Stability analysis

To study the instability, the film thickness may be written as

$$h = 1 + \eta, \tag{36}$$

where $\eta \ll 1$ are the dimensionless perturbation of the film thickness.

Setting the transformation

$$t = \varepsilon\tilde{t} \quad \text{and} \quad x = \varepsilon\tilde{x} \tag{37}$$

and using (36) and (37) in (35), and retaining the terms up to the third order fluctuations after dropping the tilde sign can be written as

$$\eta_t + A\eta_x + B\eta_{xx} + C(\mathscr{H}(\eta))_{xxx} + D\eta_{xxxx} + A'\eta\eta_x + B'(\eta\eta_x)_x + C'(\eta(\mathscr{H}(\eta))_{xx})_x + D'(\eta\eta_{xxx})_x + \frac{1}{2}A''\eta^2\eta_x + B''\left(\frac{1}{2}\eta^2\eta_x\right)_x + C''\left(\frac{1}{2}\eta^2(\mathscr{H}(\eta))_{xx}\right)_x + D'\left(\frac{1}{2}\eta^2\eta_{xxx}\right)_x + O(\eta^4) = 0, \tag{38}$$

where A, B, C, D and their corresponding derivatives are evaluated at $h = 1$.

3.1. Linear stability analysis

In this section, we are interested to study the linear response for a sinusoidal perturbation of the film by assuming the perturbation of the form

$$\eta = \Gamma[\exp\{\mathrm{i}(kx - \omega t)\}] + \text{c.c.}, \tag{39}$$

where Γ is the amplitude of the disturbance and c.c. represents complex conjugate. Here the wave number k is real and $\omega = \omega_r + \mathrm{i}\omega_i$ is the complex frequency. Using (39) in linearized part of (38), we get the dispersion relation as

$$D(\omega, k) \equiv -\mathrm{i}\omega + \mathrm{i}Ak - Bk^2 + Ck^2|k| + Dk^4 = 0. \tag{40}$$

Equating the real and imaginary parts of (40), we get

$$\omega_r = Ak \quad \text{and} \quad \omega_i = Bk^2 - Ck^2|k| - Dk^4. \tag{41}$$

Therefore, the phase speed

$$c_r = \omega_r/k = 3 - Mn, \tag{42}$$

it is to be noted here that the phase speed is independent of k, implying the wave is non-dispersive in nature. Also it is clear that linear phase speed is independent of electric Weber number but it depends on Marangoni numbers. The minimum Re at which instability sets in may be denoted as the critical Reynolds number Re_c for the wave formation and obtained from (41) as

$$Re_c = \frac{60We\cot\theta}{5E_w^2 + 3We[8(3 - Mn) + 5PrMn(5 - 2Mn)]}. \tag{43}$$

As $Mn \to 0$, $E_w \to 0$, $Re_c = (5/6)\cot\theta$, which is the critical Re for isothermal case as obtained by Benjamin [12] and Yih [13]. Also as $E_w \to 0$, the Re_c coincides as obtained by Mukhopadhyay and Mukhopadhyay [8]. Again as $Mn \to 0$, the Re_c is compatible with that of Mukhopadhyay and Dandapat [3] except the possible difference as that was calculated by momentum integral method.

In the neutral state $\omega_i = 0$ gives two relations

$$k = 0, \tag{44a}$$

$$k_c = \frac{ReE_w \pm \sqrt{Re^2E_w^2 + 36BD}}{6D}, \tag{44b}$$

which correspond to two branches of the neutral curves and the flow instability takes place in between them. Further the neutral curves intersect at the bifurcation point $Re = Re_c$, $k = 0$.

3.2. Nonlinear stability analysis

To study the growth of weakly nonlinear waves, we shall use the method of multiple scales and expand the surface elevation η as

$$\eta(x, x_1, \ldots \quad t, t_1, t_2, \ldots) = \varsigma\eta_1 + \varsigma^2\eta_2 + \varsigma^3\eta_3 + \cdots, \tag{45}$$

where the scalings $x, x_1, \ldots \quad t, t_1, t_2, \ldots$ are related according to

$$x_1 = \varsigma x, \quad t_1 = \varsigma t, \quad t_2 = \varsigma^2 t, \ldots. \tag{46}$$

Using (45) and (46) in (38) we get

$$(L_0 + \varsigma L_1 + \varsigma^2 L_2 + \cdots)(\varsigma\eta_1 + \varsigma^2\eta_2 + \varsigma^3\eta_3 + \cdots) = -\varsigma^2 N_2 - \varsigma^3 N_3 - \cdots \tag{47}$$

where L_0, L_1, L_2, etc. are the operators and N_2, N_3 are the nonlinear terms of Eq. (47) that are given in Appendix A.

In the lowest order of ς, we have

$$L_0\eta_1 = 0, \tag{48}$$

which has a solution of the form

$$\eta_1 = \Gamma(x_1, t_1, t_2)[\exp \mathrm{i}\Theta] + \text{c.c.} \tag{49}$$

where $\Theta = kx - \omega_r t$ and c.c. denotes the complex conjugates. It is to be noted here that the above solution given in (49) is already obtained in connection with the linear stability analysis except ω is replaced by ω_r, since in the vicinity of the neutral curve $\omega_i = O(\varepsilon^2)$, so that the function $\exp(\omega_i t)$ is slowly varying and may be absorbed in $\Gamma(x_1, t_1, t_2)$.

In the second order, the perturbation system yields

$$L_0\eta_2 = -L_1\eta_1 - N_2. \tag{50}$$

Invoking (49) in (50), we have

$$L_0\eta_2 = -\mathrm{i}\left[\frac{\partial D(\omega_r, k)}{\partial\omega_r}\frac{\partial\Gamma}{\partial t_1} - \frac{\partial D(\omega_r, k)}{\partial k}\frac{\partial\Gamma}{\partial x_1}\right]e^{\mathrm{i}\Theta} - \Omega\Gamma^2 e^{2\mathrm{i}\Theta} + \text{c.c.}, \tag{51}$$

where $D(\omega_r, k)$ is given by (40), and

$$\Omega = iA'k - 2B'k^2 + 2C'k^2|k| + 2D'k^4.$$

The uniform valid solution for η_2 is obtained from (50) as

$$\eta_2 = -\frac{\Omega\Gamma^2 e^{2\mathrm{i}\Theta}}{D(2\omega_r, 2k)} + \text{c.c.} \tag{52}$$

Introducing the co-ordinate transformation $\xi = (x_1 - c_g t_1)$, where $c_g (= -D_k/D_{\omega_r})$ is the group velocity, and using the solvability condition on the third order equation, we get

$$\frac{\partial\Gamma}{\partial t_2} + J_1\frac{\partial^2\Gamma}{\partial\xi^2} - \varsigma^{-2}\omega_i\Gamma + (J_2 + \mathrm{i}J_4)|\Gamma|^2\Gamma = 0, \tag{53}$$

where

$$J_1 = B - 3Ck - 6Dk^2,$$

$$J_2 = \frac{1}{2}\left(D''k^4 - 3C''k^2|k| - B''k^2\right) + \left[\frac{(A')^2k^2 - 2\left(7D'k^4 - 5C'k^2|k| - B'k^2\right)\left(D'k^4 + C'k^2|k| - B'k^2\right)}{16Dk^4 + 8Ck^2|k| - 4Bk^2}\right]$$

and

$$J_4 = \frac{1}{2}A''k - A'k\left[\frac{9D'k^4 + 7C'k^2|k| - 3B'k^2}{16Dk^4 + 8Ck^2|k| - 4Bk^2}\right].$$

For filtered waves there is no spatial modulation and the diffusion term vanishes, we get

$$\frac{\partial \Gamma}{\partial t_2} - \varsigma^{-2}\omega_i \Gamma + (J_2 + iJ_4)|\Gamma|^2\Gamma = 0. \quad (54)$$

Solution of this equation may be written as

$$\Gamma = a\, e^{-ib(t_2)t_2}, \quad (55)$$

which gives

$$\frac{\partial a}{\partial t_2} = [\varsigma^{-2}\omega_i - J_2 a^2]a, \quad (56)$$

and

$$\frac{\partial (b(t_2)t_2)}{\partial t_2} = J_4 a^2. \quad (57)$$

Eq. (56) is nothing but the Landau equation. This equation is used to characterize the nonlinear behaviour of the travelling film flow. The second term on the right-hand side of Eq. (56) is due to nonlinearity and may moderate or accelerate the exponential growth of the linear disturbance. For the existence of a supercritical stable zone in the linear unstable region ($\omega_i > 0$), the second Landau constant J_2 should be positive and the threshold amplitude will be

$$\varsigma a = [\omega_i / J_2]^{1/2}. \quad (58)$$

On the other hand, in the linear stable zone ($\omega_i < 0$) if $J_2 < 0$ the flow will be subcritically unstable and ςa is the threshold amplitude.

The nonlinear wave speed in the supercritical stable zone is obtained by

$$Nc_r = c_r + c_i(J_4/J_2), \quad \text{where} \quad c_i = \omega_i/k. \quad (59)$$

4. Side-band stability analysis

In this section, we examine whether a filtered finite-amplitude wave of film flow with phase change is stable with respect to side-band disturbance.

Spatially uniform solution of (54) in the form

$$\Gamma_\infty(t_2) = |\Gamma_\infty| \exp(-iQt_2)$$

gives

$$Q = \zeta^{-2}\omega_i \frac{J_4}{J_2}, \qquad \Gamma_\infty^2 = \frac{\zeta^{-2}\omega_i}{J_2}.$$

This solution is perturbed by spatial side-band disturbances in the form

$$\Gamma = \Gamma_\infty(t_2) + [\delta\Gamma_+(t_2)\exp(iKX) + \delta\Gamma_-(t_2)\exp(-iKX)]\exp(-iQt_2),$$

with K being the modulation wave number and it is substituted in (54). Neglecting the terms containing nonlinearities of $\delta\Gamma_+$, $\delta\Gamma_-$ one obtains

$$\frac{\partial}{\partial t_2}\begin{pmatrix}\delta\Gamma_+ \\ \delta\overline{\Gamma}_-\end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\delta\Gamma_+ \\ \delta\overline{\Gamma}_-\end{pmatrix}, \quad (60)$$

where

$$A_{11} = -2(J_2 + iJ_4)\Gamma_\infty^2 + iQ + \zeta^{-2}\omega_i + K^2 J_1,$$
$$A_{12} = -(J_2 + iJ_4)\Gamma_\infty^2, \quad A_{21} = \overline{A}_{12},$$
$$A_{22} = -2(J_2 - iJ_4)\Gamma_\infty^2 - iQ + \zeta^{-2}\omega_i + K^2 J_1.$$

Considering only the linear stability of the supercritical wave, one write the solution of (60) in the form

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix}\delta\Gamma_+ \\ \delta\Gamma_-\end{pmatrix}\exp(\lambda t_2).$$

The eigen value of λ in the above solution, after solving the eigen value problem (60) are

$$\lambda_1 = J_1 K^2, \quad \lambda_2 = J_1 K^2 - 2\zeta^{-2}\omega_i.$$

From (60) the trace of the matrix = $A_{11} + A_{22}$ is real and negative for $\omega_i > 0$, therefore at least one of the eigen value, in this case, is real and negative, which corresponds to the linearly sideband stable mode. The other eigen value can, however, have either positive or negative real part depending the values of the problem parameters. If both the eigen values are negative then the spatially uniform solution is stable with respect to the side-band disturbances for $\omega_i > 0$.

5. Results and discussion

The object of this study is to quantify the thermocapillary effect on a viscous film flowing down an inclined plane with linear temperature variation in presence of normal electric field in the finite amplitude regime. Physical parameters chosen for this investigation are: (1) Reynolds number ranging from 0 to 2. (2) Marangoni number ranging from 0 to 0.6. (3) Electric Weber number ranging from 0 to 50. (4) Prandtl number 7. (5) Weber number 450. The ongoing analysis is completely theoretical one. Since, a complete set of data regarding the necessary physical properties for a particular fluid that can perfectly match with our problem concerned are not available, therefore it is very difficult to estimate the value of the parameters involved in this problem for a particular fluid. To estimate the electric Weber number E_w we have considered a highly viscous liquid glycerine of mean film thickness $h_0 = 2.0 \times 10^{-3}$ m, kinematic viscosity $\nu = 1.19 \times 10^{-3}$ m^2/s, density $\rho = 1260$ kg/m^3, the inclination angle $\theta = \pi/3$ and a very high electric field $E_0 = 800$ kV/m (although it is very strong but is still almost one-third of the dielectric breakdown of the field in air) which gives $E_w = 49.8$. For estimation of the other parameter, like Mn we refer [8].

Fig. 1 shows the variation of Re_c with E_w for different Mn. It is clear from the figure that as Mn increases Re_c decreases showing the destabilizing effect of Mn on the other hand Re_c also decreases with E_w, confirming the destabilizing effect of E_w. Therefore, we conclude that the influence of both the parameters Mn and E_w are qualitatively same but there is a quantitative difference between the parameters. It is clear from the figure that the effect of electric field is very feeble compare to the thermocapillary effect. The reason behind the fact is that the electric parameter does not appear in the first order correction of the evolution equation (35) whereas the Marangoni number appears from the first order terms.

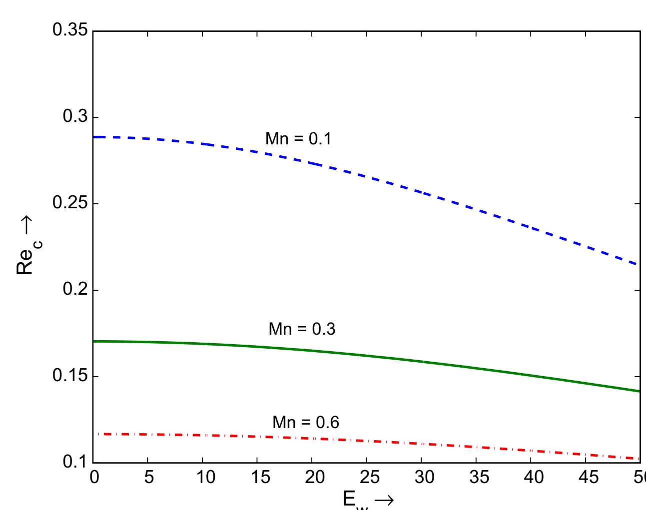

Fig. 1. Variation of Re_c for different E_w and for $\theta = \pi/3$, $We = 450$ and $Pr = 7$.

Numerically analyzing the nature of J_2 and ω_i we have found that supercritical stable, subcritical unstable, unconditional and explosive zones are possible with the variation of the Marangoni number Mn. As for example for a fixed $k = 0.145$ the variation of J_2 and ω_i with the Marangoni number Mn are depicted in Fig. 2. It reveals from the figure that for Marangoni number $Mn \leqslant 0.2472$(approx) subcritical unstable and for $Mn \geqslant 0.3007$(approx) supercritical stable zones are possible and in the intermediate range of Mn unconditional stable zone exist. Similarly for a fixed wave number and for a proper chosen parameters other than E_w it is found that all the above stability zones can exist with the variation of E_w [3]. Thus both Marangoni number Mn and electric Weber number E_w control the stability criteria.

Variation of the threshold amplitude with the Marangoni number Mn in the supercritical stable and subcritical unstable zones for a fixed wave number and for different electric parameters E_w are shown in Figs. 3 and 4, respectively. Fig. 3 reveals that the threshold amplitude increases with both Mn and E_w in the supercritical stable region, while Fig. 4 shows that in the subcritical unstable region it decreases with the increase of the parameters Mn and E_w confirming the destabilizing role of the parameters Mn and E_w. It is observed from Fig. 3 that in the supercritical stable region the threshold amplitude becomes larger rapidly for the initial change in E_w but as E_w increases the rate of increase diminishes.

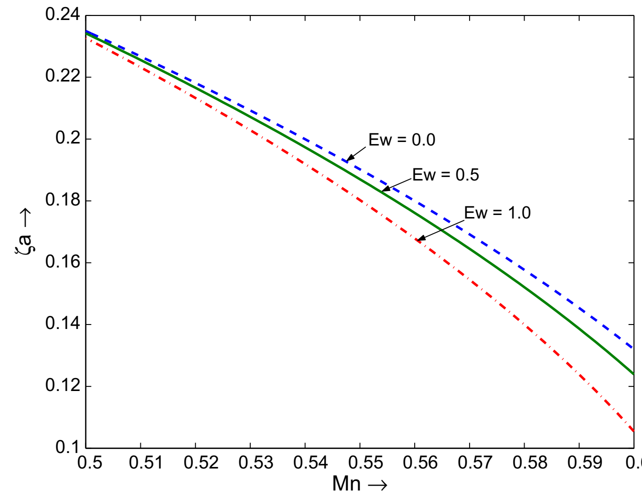

Fig. 4. Amplitude of disturbances in the subcritical region for different Mn and for $k = 0.132$, $Re = 2$, $We = 450$, $\theta = \pi/3$ and $Pr = 7$.

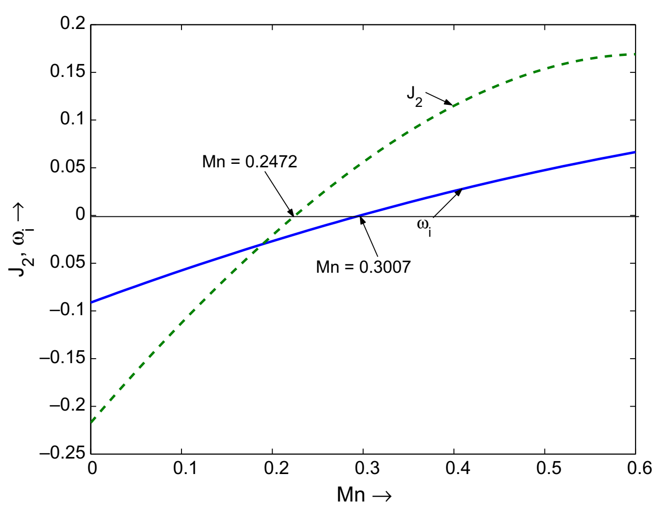

Fig. 2. Variation of J_2 and ω_i for different Mn and for fixed values of $k = 0.145$, $Re = 2$, $We = 450$, $\theta = \pi/3$, $Pr = 7$ and $E_w = 1$.

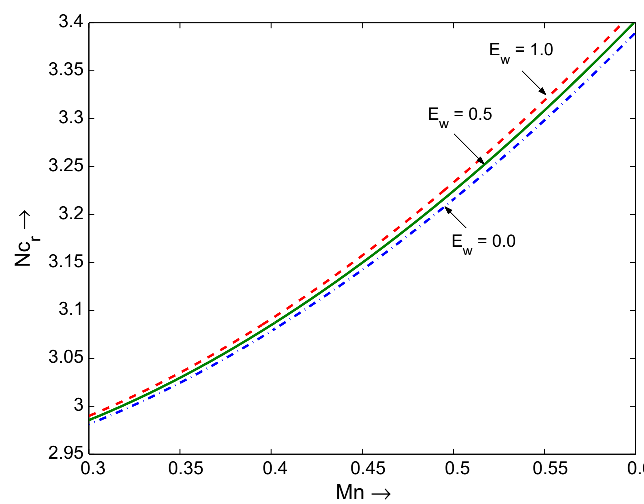

Fig. 5. Nonlinear wave speed for different Mn in the supercritical region for $k = 0.08$, $Re = 2$, $We = 450$, $\theta = \pi/3$, $Pr = 7$ and $E_w = 1$.

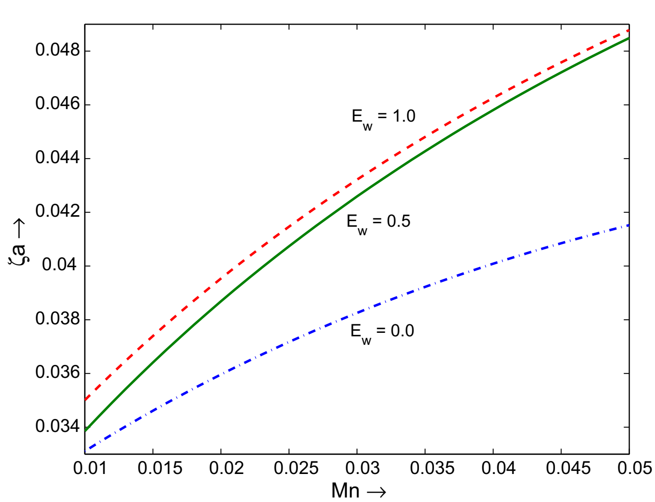

Fig. 3. Amplitude of disturbances in the supercritical region for different Mn and for $k = 0.132$, $Re = 2$, $We = 450$, $\theta = \pi/3$ and $Pr = 7$.

Fig. 5 shows the variation of nonlinear wave speed with the Marangoni number Mn in the supercritical stable region for a fixed wave number and for different values of electric parameters E_w. It is clear from the figure that nonlinear wave speed increases with both Mn and E_w as expected.

Finally, by computing the eigen value problem for the side-band instability we found that for considered parameter range both the eigen values λ_1 and λ_2 are negative, so the supercritical waves are stable with regard to side-band disturbance.

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Appendix A

$$L_0 \equiv \frac{\partial}{\partial t} + A\frac{\partial}{\partial x} + B\frac{\partial^2}{\partial x^2} + C\mathscr{H}o\frac{\partial^3}{\partial x^3} + D\frac{\partial^4}{\partial x^4},$$

$$L_1 \equiv \frac{\partial}{\partial t_1} + A\frac{\partial}{\partial x_1} + 2B\frac{\partial^2}{\partial x\partial x_1} + 3C\mathscr{H}o\frac{\partial^3}{\partial x^2\partial x_1} + 4D\frac{\partial^4}{\partial x^3\partial x_1},$$

$$L_2 \equiv \frac{\partial}{\partial t_2} + B\frac{\partial^2}{\partial x_1^2} + 3C\mathscr{H}o\frac{\partial^3}{\partial x\partial x_1^2} + 6D\frac{\partial^4}{\partial x^2\partial x_1^2},$$

$$N_2 = A'\eta_1\frac{\partial\eta_1}{\partial x} + B'\left[\eta_1\frac{\partial^2\eta_1}{\partial x^2} + \left(\frac{\partial\eta_1}{\partial x}\right)^2\right] + C'\left[\frac{\partial\eta_1}{\partial x}\mathscr{H}\left(\frac{\partial^2\eta_1}{\partial x^2}\right) + \eta_1\mathscr{H}\left(\frac{\partial^3\eta_1}{\partial x^3}\right)\right] + D'\left[\eta_1\frac{\partial^4\eta_1}{\partial x^4} + \frac{\partial\eta_1}{\partial x}\frac{\partial^3\eta_1}{\partial x^3}\right],$$

$$\begin{aligned} N_3 = {} & A'\left[\eta_1\left(\frac{\partial\eta_2}{\partial x} + \frac{\partial\eta_1}{\partial x_1}\right) + \eta_2\frac{\partial\eta_1}{\partial x}\right] + B'\left[\eta_1\left(\frac{\partial^2\eta_2}{\partial x^2} + 2\frac{\partial^2\eta_1}{\partial x\partial x_1}\right) + \eta_2\frac{\partial^2\eta_1}{\partial x^2} + 2\frac{\partial\eta_1}{\partial x}\left(\frac{\partial\eta_2}{\partial x} + \frac{\partial\eta_1}{\partial x_1}\right)\right] \\ & + C'\left[\frac{\partial\eta_1}{\partial x}\mathscr{H}\left(2\frac{\partial^2\eta_1}{\partial x\partial x_1} + \frac{\partial^2\eta_2}{\partial x^2}\right) + \left(\frac{\partial\eta_2}{\partial x} + \frac{\partial\eta_1}{\partial x_1}\right)\mathscr{H}\left(\frac{\partial^2\eta_1}{\partial x^2}\right) + \eta_1\mathscr{H}\left(\frac{\partial^3\eta_2}{\partial x^3} + 3\frac{\partial^3\eta_1}{\partial x^2\partial x_1}\right) + \eta_2\mathscr{H}\left(\frac{\partial^3\eta_1}{\partial x^3}\right)\right] \\ & + D'\left[\eta_1\left(\frac{\partial^4\eta_2}{\partial x^4} + 4\frac{\partial^4\eta_1}{\partial x^3\partial x_1}\right) + \eta_2\frac{\partial^4\eta_1}{\partial x^4} + \frac{\partial\eta_1}{\partial x}\left(\frac{\partial^3\eta_2}{\partial x^3} + 3\frac{\partial^3\eta_1}{\partial x^2\partial x_1}\right) + \frac{\partial^3\eta_1}{\partial x^3}\left(\frac{\partial\eta_2}{\partial x} + \frac{\partial\eta_1}{\partial x_1}\right)\right] + \frac{1}{2}A''\eta_1^2\frac{\partial\eta_1}{\partial x} + B''\left[\frac{1}{2}\eta_1^2\frac{\partial^2\eta_1}{\partial x^2} + \eta_1\left(\frac{\partial\eta_1}{\partial x}\right)^2\right] \\ & + C''\eta_1\left[2\frac{\partial\eta_1}{\partial x}\mathscr{H}\left(\frac{\partial^2\eta_1}{\partial x^2}\right) + \eta_1\mathscr{H}\left(\frac{\partial^3\eta_1}{\partial x^3}\right)\right] + D''\left[\frac{1}{2}\eta_1^2\frac{\partial^4\eta_1}{\partial x^4} + \eta_1\frac{\partial\eta_1}{\partial x}\frac{\partial^3\eta_1}{\partial x^3}\right]. \end{aligned}$$

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